Project systems theory

22/01/2015, Thursday, 08:30-11:30

You are **NOT** allowed to use any type of calculators.

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1 (10 + 10 = 20 pts) **Linearization**

A simplified model of a patient in the presence of an infectious disease is described by the equations

$$\dot{\xi} = 1 - \xi - \alpha\xi\eta \tag{1a}$$
$$\dot{\eta} = \alpha\xi\eta - \eta - \beta\eta \tag{1b}$$

in which ξ represents the number of non-infected cells, η represents the number of infected cells, α represents the effect of the therapy and $\beta \in (0, 1)$ represents the action of the immune system.

1. Consider a nonlinear system of the form $\dot{x}(t) = f(x(t))$. A (constant) vector $\bar{x}$ is called an *equilibrium point* if $f(\bar{x}) = 0$. Determine the two equilibrium points of the system (1). Show that one equilibrium corresponds to a healthy patient, i.e. the number of infected cells is zero, and one equilibrium corresponds to an ill patient, i.e. the number of infected cells is non-zero.

2. Write the linearized models of the system (1) around the two equilibrium points.

Required Knowledge: **Linearization.**

Solution:

(1a): The equilibrium points can be found by solving the following two equations:

$$0 = 1 - \bar{\xi} - \alpha\bar{\xi}\bar{\eta} \tag{2a}$$
$$0 = \alpha\bar{\xi}\bar{\eta} - \bar{\eta} - \beta\bar{\eta} \tag{2b}$$

It follows from (2b) that

$$\bar{\eta} = 0 \quad \text{or} \quad \bar{\xi} = \frac{1+\beta}{\alpha}$$

where $\alpha \neq 0$. Substituting these two possibilities in (2a), we obtain two equilibrium points:

$$(\bar{\xi}, \bar{\eta}) = (1, 0) \quad \text{or} \quad (\bar{\xi}, \bar{\eta}) = \left(\frac{1+\beta}{\alpha}, \frac{-1+\alpha-\beta}{\alpha(1+\beta)}\right).$$

The first one represents healthy patients as $\bar{\eta} = 0$ whereas the second ill patients as $\bar{\eta} > 0$.

(1b): Let

$$f_1(\xi, \eta) = 1 - \xi - \alpha\xi\eta \quad \text{and} \qquad f_2(\xi, \eta) = \alpha\xi\eta - \eta - \beta\eta.$$

Note that

$$\frac{\partial f_1}{\partial \xi} = -1 - \alpha\eta, \quad \frac{\partial f_1}{\partial \eta} = -\alpha\xi, \quad \frac{\partial f_2}{\partial \xi} = \alpha\eta, \quad \frac{\partial f_2}{\partial \eta} = \alpha\xi - 1 - \beta.$$

Then, the linearization around the two equilibria found above can be given by

$$\dot{x} = A_i x$$

where

$$A_1 = \begin{bmatrix} -1-\alpha\eta & -\alpha\xi \\ \alpha\eta & \alpha\xi - 1 - \beta \end{bmatrix}_{(\xi,\eta)=(1,0)} = \begin{bmatrix} -1 & -\alpha \\ 0 & -1+\alpha-\beta \end{bmatrix}$$

$$A_2 = \begin{bmatrix} -1-\alpha\eta & -\alpha\xi \\ \alpha\eta & \alpha\xi - 1 - \beta \end{bmatrix}_{(\xi,\eta)=(\frac{1+\beta}{\alpha}, \frac{-1+\alpha-\beta}{\alpha(1+\beta)})} = \begin{bmatrix} \dfrac{-\alpha}{1+\beta} & -1-\beta \\ \dfrac{-1+\alpha-\beta}{1+\beta} & 0 \end{bmatrix}.$$

Let

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -a & -a & -a & -2 \end{bmatrix}.$$

Determine all values of a such that the system $\dot{x} = Ax$ stable?

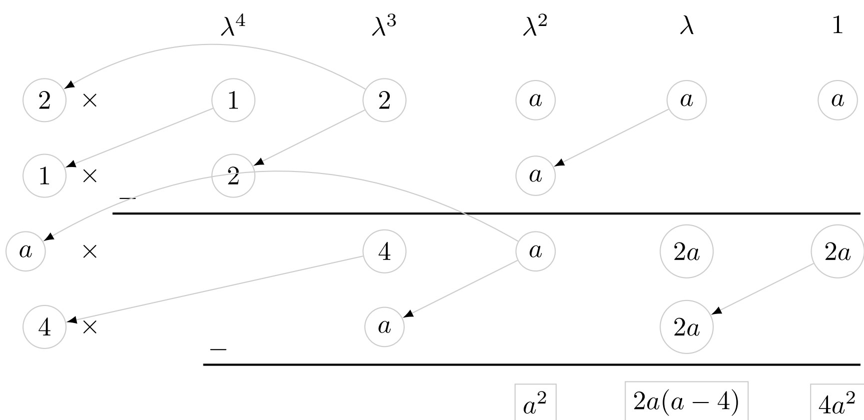
Required Knowledge: **Routh-Hurwitz criterion, companion form.**

Solution:

As the matrix A is in the so-called companion form, its characteristic polynomial can be found as

$$p_A(\lambda) = \lambda^4 + 2\lambda^3 + a\lambda^2 + a\lambda + a.$$

Applying Routh-Hurwitz criterion, we get the following table:



After the first step, we see that the system is stable if and only if $a > 0$ and the polynomial $4\lambda^3 + a\lambda^2 + 2a\lambda + 2a$ is stable. After the second step, we see that the system is stable if and only if $a > 0$ and $a - 4 > 0$. This follows from the fact that a quadratic polynomial is stable if and only if all the coefficient have the same sign. Therefore, the system is stable if and only if $a > 4$.

Consider the system $\dot{x} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} x + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u.$

1. Is it controllable?
2. Find a nonsingular T such that

$$T^{-1}AT = \begin{bmatrix} 0 & 1 \\ \alpha & \beta \end{bmatrix} \qquad T^{-1}b = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

   for some real numbers α and β.

3. By using the matrix T of the previous subproblem, find a state feedback of the form $u = k^T x$ such that the closed loop system has poles at -1 and -2.

REQUIRED KNOWLEDGE: **Controllability and pole placement**

SOLUTION:

(3a): Let the system be of the form $\dot{x} = Ax + bu$. Note that

$$\begin{bmatrix} b & Ab \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 1 & 7 \end{bmatrix}$$

is of rank 2. As such, the system is controllable.

(3b): Note that

$$\Delta_A(\lambda) = \det(\lambda I - A) = \begin{vmatrix} \lambda - 1 & -2 \\ -3 & \lambda - 4 \end{vmatrix} = \lambda^2 - 5\lambda - 2.$$

Define

$$q_2 = b = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$q_1 = Ab + a_1 b = \begin{bmatrix} 3 \\ 7 \end{bmatrix} - 5 \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 \\ 2 \end{bmatrix}.$$

Let

$$T = \begin{bmatrix} q_1 & q_2 \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ 2 & 1 \end{bmatrix}.$$

This leads to

$$T^{-1} = \frac{1}{4} \begin{bmatrix} -1 & 1 \\ 2 & 2 \end{bmatrix}$$

Note that

$$AT = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} -2 & 1 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 2 & 7 \end{bmatrix}$$

$$T^{-1}AT = \frac{1}{4} \begin{bmatrix} -1 & 1 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ 2 & 7 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 2 & 5 \end{bmatrix}$$

and

$$T^{-1}b = \frac{1}{4} \begin{bmatrix} -1 & 1 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

(3c): Let $\hat{A} = T^{-1}AT$ and $\hat{b} = T^{-1}b$. Also let $\hat{k}^T = \begin{bmatrix} k_1 & k_2 \end{bmatrix}$. Note that

$$\Delta_{\hat{A}+\hat{b}\hat{k}^T}(\lambda) = \lambda^2 - (5 + k_2)\lambda - (2 + k_1).$$

As we want to assign the poles at -1 and -2, we need to solve

$$\lambda^2 - (5 + k_2)\lambda - (2 + k_1) = (\lambda + 1)(\lambda + 2) = \lambda^2 + 3\lambda + 2.$$

This results in

$$k_1 = -4 \quad \text{and} \quad k_2 = -8.$$

Then, we get

$$k^T = \hat{k}^T T^{-1} = \frac{1}{4}\begin{bmatrix}-4 & -8\end{bmatrix}\begin{bmatrix}-1 & 1\\ 2 & 2\end{bmatrix} = \begin{bmatrix}-1 & -2\end{bmatrix}\begin{bmatrix}-1 & 1\\ 2 & 2\end{bmatrix} = \begin{bmatrix}-3 & -5\end{bmatrix}.$$

Consider the system Σ given by the equations

$$\dot{x}(t) = Ax(t) + Bu(t)$$
$$y(t) = Cx(t) + Du(t).$$

with $x(t) \in \mathbb{R}^n$ and $y(t) \in \mathbb{R}^p$, is called *output-controllable* if for any $x_0 \in \mathbb{R}^n$ and $y_1 \in \mathbb{R}^p$ there exists an input u and a positive real number T such that $y_u(T, x_0) = y_1$. Show that Σ is output-controllable if and only if

$$\operatorname{rank}\begin{pmatrix} D & CB & CAB & \cdots & CA^{n-1}B \end{pmatrix} = p.$$

REQUIRED KNOWLEDGE: **Controllability and solution of linear differential equations**

SOLUTION:

Note that

$$y_u(T, x_0) = Ce^{AT}x_0 + \int_0^T Ce^{A(T-\tau)}Bu(\tau)\,d\tau + Du(T).$$

This means that the system is controllable if and only if

$$\mathcal{V}_T := \{\int_0^T Ce^{A(T-\tau)}Bu(\tau)\,d\tau + Du(T) \mid u \text{ is integrable}\} = \mathbb{R}^p.$$

Now, it is enough to show that

$$\eta \in \mathcal{V}_T^\perp \qquad \text{if and only if} \qquad \eta^T D = 0 \quad \text{and} \quad \eta^T CA^k B = 0 \text{ for } k = 0, 1, \ldots, n-1.$$

For the 'if' part, let η be such that $\eta^T D = 0$ and $\eta^T CA^k B = 0$ for all $k = 0, 1, \ldots, n-1$. Note that we get

$$\eta^T Ce^{At} = 0 \quad \text{for all } t \geqslant 0$$

from the latter. Thus, we have

$$\eta^T \big(\int_0^T Ce^{A(T-\tau)}Bu(\tau)\,d\tau + Du(T) \big) = 0$$

for any integrable input u. Therefore, $\eta \in \mathcal{V}_T^\perp$.

For the 'only if' part, let $\eta \in \mathcal{V}_T^\perp$. First, we claim that

$$\eta^T D = 0. \tag{3}$$

To prove this claim, for $\varepsilon > 0$ and arbitrary $\bar{u}$ define

$$u_\varepsilon(t) = \begin{cases} 0 & \text{if} \quad 0 \leqslant t < T - \varepsilon \\ \bar{u} & \text{if} \quad T - \varepsilon \leqslant t \leqslant T. \end{cases}$$

Since $\eta \in \mathcal{V}_T^\perp$, we have

$$\eta^T D\bar{u} = -\eta^T \int_0^T Ce^{A(T-\tau)}Bu_\varepsilon(\tau)\,d\tau$$

for all $\bar{u}$ and $\varepsilon > 0$. By taking the limit as ε tends to zero, we obtain

$$\eta^T D\bar{u} = 0$$

since

$$\lim_{\varepsilon \to 0} \int_0^T Ce^{A(t-\tau)}Bu_\varepsilon(\tau)\,d\tau = 0.$$

As $\bar{u}$ arbitrary, we can conclude that (3) holds. Further, we have

$$\eta^T \int_0^T Ce^{A(T-\tau)}Bu(\tau)\,d\tau = 0$$

for any integrable u. By choosing

$$u(\tau) = B^T e^{A^T(t-\tau)}C^T\eta,$$

we get

$$0 = \eta^T \int_0^T Ce^{A(T-\tau)}Bu(\tau)\,d\tau = \int_0^T \|B^T e^{A^T(t-\tau)}C^T\eta\|^2\,d\tau.$$

This yields that

$$\eta^T Ce^{At}B = 0$$

for all t with $0 \leqslant t \leqslant T$. By differentiating and evaluating at $t = 0$, we obtain

$$\eta^T CA^k B = 0$$

for all $k = 0, 1, \ldots, n-1$.